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AroniSmArtLytics™

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1 Welcome to AroniSmartLytics

AroniSmartLytics is a quick statistical reference tool. It is intended for beginners, students of statistics, casual and regular users and advanced statisticians. We recommend that you refer to this manual to familiarize yourself with the major probability distributions. You may also wish to search and jump to any section with the distribution of interest. Remember to look at the relationships among common probability distributions as displayed in the attached panel. The following sections cover these probability distributions:

- **Discrete**: Bernoulli, Binomial, Discrete Uniform, Geometric, Hypergeometric, Negative Binomial, Poisson.
- **Continuous**: Beta, Cauchy, Chi Squared, Double Exponential, Exponential, Fisher-Snedecor's F, Gamma, Logistic, Lognormal, Normal, Pareto, Student's t, Uniform, Weibull.

A discrete probability function is a statistical distribution whose variables can take on only discrete values. It is described by a **probability mass function** (**pmf**), which gives the probability that a discrete random variable, *X*, is exactly equal to some value. The probability mass function exists for either scalar or discrete multivariate random variables.

A continuous probability function is described by a **probability density function** (**pdf**), which gives the likelihood that a continuous random variable, *X*, occurs at a given point. Unlike the probability mass function, the probability for the continuous random variable to fall within a particular region is given by the integral of this variable's density over the region. The probability density function is nonnegative everywhere, and its integral over the entire space of possible values is equal to one.

For each probability distribution the following concepts or statistics will be described:

cdf: cumulative distribution function describes the probability that a real-valued random variable X with a given probability distribution will be found at a value less than or equal to x.

Mgf: moment generating function is an alternative definition of the probability distribution and provides the basis of a route to closed form analytical results, compared with working directly with probability density or mass functions or cumulative distribution functions.

Mean is the weighted average of all possible values of a random variable. The weights used in computing this average correspond to the probabilities in case of a discrete random variable, or densities in case of a continuous random variable.

Variance is a measure of how far a set of numbers is spread out from each other or how far the numbers are spread from the mean.

Mode is the value that occurs most frequently in the probability distribution.

Median is the numerical value separating the higher half of a probability distribution from the lower half.

Skewedness measures the asymmetry of the probability distribution of a real-valued random variable. The skewedness value can be positive or negative, or even undefined

Kurtosis measures the "peakedness" of the probability distribution of a real-valued random variable. The Kurtosis, although measuring the peakedness, describes the degree of how heavy the distribution tails are. Higher kurtosis means more of the variance is the result of infrequent extreme deviations, or similarly heavy tails.

2 Discrete Distribution

2.1 Bernoulli distribution

The **Bernoulli distribution**, named after the Swiss scientist Jacob Bernoulli is a discrete probability distribution. Denoted as *Bernoulli (p),* it takes the value 1 with the success probability p and value 0 with the failure probability $q = 1 - p$.

The parameter of Bernoulli distribution is p.

Statistic	Formula	Comment
pmf	$P(X = x\|p = p^x(1-p)^{1-x}$; $x = 0,1;\ 0 \le p \le 1$	$\Pr(x=1)$ $= 1 - \Pr(X=0)$ $= 1 - q$ $= p$
cdf	$P(X \le x = \begin{cases} 0 & for\ x < 0 \\ q & for\ 0 \le x < 1 \\ 1 & for\ x \ge 1 \end{cases}$	$x = \{0; 1\}$
mean	p	$0 \le p \le 1$
variance	$p(1-p)$	pq
median	N/A	Not Available
mode	$\begin{cases} 0 & if\ q > p \\ 0,1 & if\ q = p \\ 1 & if\ q < p \end{cases}$	
mgf	$M_x(t) = (1-p) + pe^t$	$q + pe^t$
skewness	$\frac{q-p}{\sqrt{pq}}$	$pq = p(1-q)$
kurtosis	$\frac{6p^2 - 6p + 1}{p(1-p)}$	$\frac{6p^2 - 6p + 1}{pq}$

2.2 Binomial distribution

The **Binomial Distribution** is the discrete probability distribution that models the number of successes in a sequence of ***n*** independent yes/no or failure/success experiments, each of which yields success with probability ***p***. Denoted as ***B(n,p)*** or ***Binomial (n,p)***, the sequence of a success/failure experiment is also called a Bernoulli experiment or Bernoulli trial. The parameters of a Binomial distribution are ***n*** and ***p***

The table of relationships depicts the relationship between Binomial distribution with several other distributions including ***Bernoulli, Hypergeometric, and Normal***. The binomial distribution forms the basis for the popular binomial test of statistical significance.

Statistic	Formula	Comment
pmf	$P(X = x\|n,p = \binom{n}{x} p^x (1-p)^{(n-x)}$; $x = 0,1,2,\cdots,n$; $0 \leq p \leq 1$	$\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$
cdf	$P(X \leq x) = \sum_{k=0}^{x} \binom{n}{x} p^k (1-p)^{n-k}$ $= (n-k)\binom{n}{x} \int_0^{1-p} t^{n-k-1}(1-t)^k dt$	$= I_{1-p}(n-k, k+1)$ is regularized incomplete beta function. $k = \{0,1,\cdots,n\}$
mean	np	$n(1-q)$
variance	$np(1-p)$	npq
median	$\lfloor np \rfloor$ or $\lceil np \rceil$	
mode	$\lfloor (n+1)p \rfloor$ *or* $\lfloor (n+1)p - 1 \rfloor$	
mgf	$M_x(t) = [pe^t + (1-p)]^n$	$[(pe^t + q)]^n$
skewness	$\frac{1-2p}{\sqrt{npq}}$	$pq = p(1-q)$
kurtosis	$\frac{6p^2 - 6p + 1}{np(1-p)}$	$= \frac{1-6p(1-p)}{npq}$

2.3 Uniform Discrete Distribution

The **Uniform Discrete Distribution** is the discrete probability distribution that models the finite number of equally spaced values that are equally likely to be observed in independent observations. The number of values is usually denoted ***n***, and every one of the n values has the probability ***1/n*** of being observed.
A uniform discrete random variable has any of *n* possible values, $\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}, \cdots, \boldsymbol{x_n}$ with an equal probability of outcome, $1/_{n}$. The parameters of Uniform discrete distribution are ***n***, ***a***, and ***b***, with ***a*** being the lowest value and ***b*** the largest value of the range.

Statistic	Formula	Comment
pmf	$f(x) = \begin{cases} \frac{1}{n}, & for\ a \le x \le b \\ 0, & x > b; x < a \end{cases}$; $b \ge a;\ x = a, a+1, \cdots, b-1, b$	$a = \{\cdots, -1, 0, 1, \cdots\}$ $b = \{\cdots, -1, 0, 1, \cdots\}$ $n = b - a + 1$
cdf	$\Pr(X \le x) = \begin{cases} 0 & for\ \ k < a \\ \frac{\lfloor k \rfloor - a + 1}{n} & for\ a \le x \le b \\ 1 & for\ \ k > b \end{cases}$	
mean	$\frac{a+b}{2}$	
variance	$\frac{(b-a+1)^2 - 1}{12}$	$= \frac{n^2\ - 1}{12}$
median	$\frac{a+b}{2}$	
mode	N/A	
mgf	$M_x(t) = 1/_{N} \sum_{i=1}^{N} e^{it}$	$=\frac{e^{at} - e^{(b+1)t}}{n(1-e^t)}$
skewness	0	
kurtosis	$-\frac{6(n^2+1)}{5(n^2-1)}$	

2.4 Geometric

The **Geometric Distribution** is the discrete probability distribution that models the number of failures in a sequence of independent yes/no or failure/success experiments, before the first success, each experiment having the probability ***p*** of yielding a success. Hence, the Geometric distribution may be defined in two ways, both ways having the same meaning and leading to the same results:

- The probability distribution of the number X of Bernoulli trials needed to get one success. It this case, the set of trials is $\{1, 2, 3, \ldots\}$
- The probability distribution of the number $Y = X - 1$ of failures before the first success. In this case, the set of trials describing Y is $\{0, 1, 2, 3, \ldots\}$

The first case is usually the common convention. The second is a **Negative Binomial** distribution with parameters ***(1, p)***. Denoted as: **Geometric*(p)*,** the sequence of success/failure experiments, before yielding the first success is memoryless, $P(X > s|X > t) = P(X > s - t)$. It is described by one parameter ***p***, the probability of success at each trial.

Statistic	Formula: Case 1: X	Case 2: Y=(X-1)
pmf	$P(X = x\|p) = p(1-p)^{(x-1)}$; $x = 1, 2, \cdots$; $0 \le p \le 1$	$P(X = x\|p) = p(1-p)^{(x)}$; $x = 0,1, 2, \cdots$; $0 \le p \le 1$
cdf	$P(X \le x) = 1 - (1-p)^x$	$1 - (1-p)^{x+1}$
mean	$\frac{1}{p}$	$\frac{1-p}{p}$
variance	$\frac{1-p}{p^2}$	$\frac{1-p}{p^2}$
median	$\left\lceil \frac{-1}{log_2(1-p)} \right\rceil$; $not\ unique\ if$ $-\frac{1}{log_2(1-p)}$ $is\ an\ integer$	$\left\lceil \frac{-1}{log_2(1-p)} \right\rceil - 1$
mode	1	0
mgf	$\frac{pe^t}{1-(1-p)e^t}; t < -\log(1-p)$	$\frac{p}{1-(1-p)e^t}$
skewness	$\frac{2-p}{\sqrt{1-p}}$	$\frac{2-p}{\sqrt{1-p}}$
kurtosis	$6 + \frac{p^2}{1-p}$	$6 + \frac{p^2}{1-p}$

2.5 Hypergeometric

The **Hypergeometric Distribution** is the discrete probability distribution that models the number of successes in a sequence of ***K*** yes/no or failure/success experiments, from a finite population, **without replacement**. Hypergeometric is described by three parameters: Population size, ***N***, the number of possible successes within the population, ***M*** and the sample size ***K***.
Hence, the Geometric distribution may be described by the following table, assuming "***green***" is the successful draw:

	Drawn	Not Drawn	Total
Green	x	$M - x$	M
Other	$K - x$	$N + x - K - M$	$N - M$
Total	K	$N - K$	N

Statistic	Formula: X	Comment: Y=(X-1)
pmf	$P(X = x\|N, M, K = \frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}$; $x \in \{\max(0, K + M - N), \cdots, \min(M, N)\}$ $M - (N - K) \leq x \leq M; N, M, K \geq 0$	$N \in \{1,2,\cdots\}$ $M \in \{0,1,2,\cdots,N\}$ $K \in \{1,2,\cdots,N\}$
cdf	$P(X \leq x) = 1 - (1 - p)^x$	$1 - (1 - p)^{x+1}$
mean	$\frac{KM}{N}$	$\frac{KM}{p}$
variance	$\frac{KM}{N}\frac{(N - M)(N - K)}{N(N - 1)}$	
median		
mode	$\left\lfloor\frac{(K + 1)(M + 1)}{N + 2}\right\rfloor$	*Floor value*
mgf		
skewness	$\frac{(N - 2M)(N - 1)^{\frac{1}{2}}(N - 2K)}{[KM(N - M)(N - K)]^{\frac{1}{2}}(N - 2)}$	
kurtosis	$\frac{[(N - 1)N^2(N(N + 1) - 6M(N - M) - 6K(N - K)) + 6KM(N - M)(5N - 6)]}{KM(N - M)(N - K)(N - 2)(N - 3)}$	

2.6 Negative Binomial

The **Negative Binomial Distribution** is the discrete probability distribution that models the number of successes in a sequence of ***Bernouilli trials*** (yes/no or failure/success experiments), until a specified number of failures, ***r***, happen.

Hence, the Negative Binomial distribution may be described as the number of successes, until the specified ***rth*** failure occurs. The ***Negative Binomial*** is described by two parameters: the **number of failures, r** and the **probability of success, p**.

Some special cases of Negative Binomial distribution are known as **Pascal distribution** (named after Blaise Pascal) or Polya distribution (names after George Polya).

Statistic	Formula	Comment
pmf	$P(X = x \vert r, p = \binom{r+x-1}{x} p^r (1-p)^{(x)}$; $x = 0, 1, 2, \cdots$; $0 \leq p \leq 1$	$\binom{r+x-1}{x} = \frac{(r+x-1)!}{x!\,(r-1)!}$
cdf	$P(X \leq x) = \sum_{k=0}^{x} \binom{r+x-1}{x} p^r (1-p)^x$	$= 1 - I_p(x+1, r)$ is regularized incomplete beta function.
mean	$\frac{r(1-p)}{p}$	
variance	$Var\ X = \frac{r(1-p)}{p^2}$	
median		
mode		
mgf	$M_x(t) = [\frac{p}{1-(1-p)e^t}]^r$	$t < -log(1-p)$
skewness	$\frac{2-p}{\sqrt{(1-p)r}}$	
kurtosis	$\frac{p^2-6p+6}{r(1-p)}$	

2.7 Poisson

The **Poisson Distribution** is the discrete probability distribution that models a number of events occurring in a fixed interval of time, space, area, or volume, given that these events occur with a known average rate, λ and independently of the time since the last event.

The Poisson was named after the French Mathematician Simeon Denis Poisson.

Statistic	Formula	Comment
pmf	$P(X = x\|\lambda = \frac{\lambda^{x}}{x!}\ e^{-\lambda};$ $x = 0, 1, 2, \cdots\ ;\ \lambda \geq 0$	
cdf	$\Pr(X \leq x) = e^{-\lambda} \sum_{i=0}^{x} \frac{\lambda^i}{i!}$	$\frac{\Gamma(\lfloor x+1 \rfloor, \lambda)}{\lfloor k \rfloor !})$ for $x \geq 0$.
mean	λ	
variance	λ	
median	$\approx \left\lfloor \lambda + \frac{1}{3} - \frac{0.02}{\lambda} \right\rfloor$	Floor value
mode	$\lceil \lambda \rceil - 1$	
mgf	$M_x(t) = e^{\lambda(e^t - 1)}$	
skewness	$\lambda^{-1/2}$	
kurtosis	λ^{-1}	

3 Continuous Distribution

3.1 Beta Distribution

The **Beta Distribution** is a family of continuous probability distribution that models an unknown probability variable. It is used to model probabilities such as the probability of success in a Bernoulli or Binomial distribution. The Beta distribution is described by two positive parameters: $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$**.** The parameters are known as shape parameters.

Statistic	Formula	Comment
pdf	$P(X = x\|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1};$ $0 \le x \le 1;\ \alpha > 0;\ \beta > 0$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(a)\Gamma(b)} = \frac{1}{B(\alpha,\beta)}$
cdf	$I_x(\alpha,\beta)$ Bx(α,β) is the incomplete Beta Function and *I*x(α,β) is the regularized Beta Function	$I_x(\alpha,\beta)$ $= \frac{B_x(\alpha,\beta)}{B(\alpha,\beta)}$
mean	$\frac{\alpha}{\alpha+\beta}$	
variance	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
median	$I_{0.5}^{-1}(\alpha,\beta)$	No closed form
mode	$\frac{\alpha-1}{(\alpha+\beta-2)}$	$\alpha > 1, \beta > 1$
mgf	$M_x(t) = 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t}{k!}^{k}$	
skewness	$\frac{2(\beta-\alpha)\sqrt{\alpha+\beta+1)}}{(\alpha+\beta+2)\sqrt{\alpha\beta}}$	
Kurtosis-excess	$\frac{6[\alpha^3 - \alpha^2(2\beta-1)+\beta^2(\beta+1)-2\alpha\beta(\beta+2)}{\alpha\beta(\alpha+\beta+2)(\alpha+\beta+3)}$	

3.2 Cauchy Distribution

The **Cauchy Distribution,** also known as Cauchy-Lorentz distribution, named after the French mathematician Baron Augustin-Louis Cauchy and the Dutch physicist Hendrik Antoon Lorentz, is continuous probability distribution that models several physical processes and mathematical solutions. It is described by a location parameter, $\boldsymbol{\theta}$, and a scale parameter, $\boldsymbol{\sigma}$. The location parameter specifies the location of the mode of the distribution, whereas the scale parameter the half-width at half-maximum, which is equally the interquartile range, also known as **probable error**.

Statistic	Formula	Comment
pdf	$P(X=x\|\theta,\sigma)=\dfrac{1}{\pi\sigma\left[1+\left(\frac{x-\theta}{\sigma}\right)^2\right]};$ $0<x<\infty;\ \sigma>0$	
cdf	$\frac{1}{\pi}\,\text{arctan}\left(\frac{x-\theta}{\sigma}\right)+\frac{1}{2}$	
mean	$does\ not\ exist$	
variance	$does\ not\ exist$	
median	θ	
mode	θ	
mgf	$does\ not\ exist$	
skewness	$does\ not\ exist$	
Kurtosis-excess	$does\ not\ exist$	

3.3 Chi Squared Distribution

The **Chi Squared Distribution** with ν (nu) dgrees of freedom is a continuous distribution that models a sum of the squares of ν independent standard random variables. It is a special case of the Gamma distribution. It is widely used in statistical inference, especially in hypothesis testing, construction of confidence intervals, and non-parametric statistics.

Statistic	Formula	Comment
pdf	$P(X = x\|\nu) = \frac{1}{\Gamma(\frac{\nu}{2})2^{(\frac{\nu}{2})}} x^{(k/2)-1} e^{-\frac{x}{2}};$ $0 \leq x < \infty;\ \nu = 1,2,\cdots$	
cdf	$= \frac{1}{\Gamma(\frac{\nu}{2})} \gamma\left({}^{x}/_{2}, {}^{x}/_{2}\right)$ $\gamma\left({}^{x}/_{2}, {}^{x}/_{2}\right)$ is a lower incomplete gamma function.	$= P\left({}^{x}/_{2}, {}^{x}/_{2}\right)$ (Regularized Gamma Function)
mean	ν	
variance	2ν	
median		
mode	$\max(\nu - 2,0)$	
mgf	$M_x(t) = (1 - 2t)^{\nu/2}$	for t < 1/2
skewness	$\sqrt{{}^{8}/_{k}}$	
Kurtosis-excess	${}^{12}/_{k}$	

3.4 Double Exponential Distribution

The **Double Exponential Distribution** usually refers to two distributions: Laplace distribution and Gumbel distribution.
Laplace distribution, described below is also known as bilateral exponential distribution, and consists of two exponential distributions glued together on each side of a threshold. It was named after the French mathematician Pierre-Simon Marquis de Laplace.
Gumbel distribution, named after the German mathematician Emil Julius Gumbel, is a cumulative distribution function, which is the exponent of an exponent.
The Laplace double exponential distribution is a continuous probability distribution. It is described by two parameters: a **location parameter,** $\boldsymbol{\mu}$ and **a scale parameter,** $\boldsymbol{\sigma}$**.**

Statistic	Formula	Comment
pdf	$P(X = x\|\mu,\sigma) = \frac{1}{2\sigma} e^{\frac{-\|x-\mu\|}{\sigma}}$; $-\infty < x < \infty;\ -\infty < \mu < \infty;\ \sigma > 0$	
cdf	$= 0.5[1 + sign(x-\mu)(1 - \exp(-\|x-\mu\|/\sigma))]$	
mean	μ	
variance	$2\sigma^2$	
median	μ	
mode	μ	
mgf	$M_x(t) = \frac{\exp(\mu t)}{1-\sigma^2 t^2}$ for |t| < $1/\sigma$	
skewness	0	
Kurtosis-excess	3	

3.5 Exponential Distribution

The **Exponential Distribution,** sometimes known as **negative exponential** distribution is a family of continuous probability distributions that models the time between two events in a Poisson process. The Poisson process is a process in which discrete independent events occur continuously at a constant average rate.

The exponential distribution is described by one parameter: λ, the inverse of the average rate, or the average time between two events.

Statistic	Formula	Comment
pdf	$P(X = x\|\lambda) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}$; $0 \leq x < \infty; \lambda > 0$	
cdf	$1 - e^{\frac{-x}{\lambda}}$	
mean	λ	
variance	λ^2	
median	$\lambda \ln 2$	
mode	0	
mgf	$M_x(t) = \frac{1}{1-\lambda t}$ for t < $\frac{1}{\lambda}$	
skewness	2	
Kurtosis-excess	6	

3.6 F Distribution

The **F Distribution** with ν_1, ν_2 degrees of freedom is a continuous distribution that models the ratio of two Chi Squared distributions scaled by their degrees of freedom. It is also often referred to as Snedecor's F distribution or the Fisher-Snedecor distribution, after the English statistician Sir Ronald Aylmer Fisher and the American mathematical statistician George Waddel Snedecor. It is described by two parameters, ν_1, ν_2 the number of degrees of freedom.

Statistic	Formula
pdf	$P(X=x\vert\nu_1,\nu_2) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}\frac{x^{(\nu_1-2)/2}}{\left(1+\left(\frac{\nu_1}{\nu_2}\right)x\right)^{(\nu_1-\nu_2)/2}};$ $0 \le x < \infty;\ \nu_1,\nu_2 = 1,2,\cdots$
cdf	$= I_{\frac{\nu_1 x}{\nu_1 x+\nu_2}}\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)$
mean	$\frac{\nu_2}{\nu_2-2}\ \ for\ \ \nu_2 > 2$
variance	$\frac{2{\nu_2}^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)(\nu_2-2)^2}\ for\ \ \nu_2 > 4$
median	
mode	$\frac{\nu_1-2}{\nu_1}\frac{\nu_2}{\nu_2+2}\ \ for\ \nu_1 > 2$
mgf	
skewness	$\frac{\sqrt{8(\nu_2-4)}\ (2\nu_1+\nu_2-2)}{(\nu_2-6)\sqrt{\nu_1(\nu_1+\nu_2-2)}}\ for\ \ \nu_2 > 6$
Kurtosis-excess	$\frac{20\nu_2 -\ 8{\nu_2}^2 +\ {\nu_2}^3 + 44\nu_1 - 33\nu_1\nu_2 + K)}{\nu_1(\nu_2-6)\ (\nu_2-8)\ (\nu_1+\nu_2-2)\ /12}\ for\ \ \nu_2 > 8$

Where K = $5{\nu_2}^2\nu_1 - 22{\nu_1}^2 - 5\nu_2{\nu_1}^2 - 16$

3.7 Gamma Distribution

The **Gamma Distribution** is a family of continuous probability distributions that usually models the waiting time until an event happens. It is described by two parameters: the **shape parameter, $\boldsymbol{r}$** and the **scale parameter, $\boldsymbol{\lambda}$**. When the shape parameter, r, is an integer, then the distribution represents an **Erlang** Distribution, named after the Danish mathematical statistician Agner Krarup Erlang, which is a sum of k independent exponential distributions, each with the parameter λ, the inverse of the average rate, or the average time between two events.

Statistic	Formula	Comment
pdf	$P(X = x\|r,\lambda) = \frac{1}{\Gamma(\mathrm{r})\lambda^r}x^{r-1}e^{-\frac{x}{\lambda}};$ $0 \le x < \infty; \quad r,\lambda > 0$	
cdf	$\frac{\gamma(r,x,\theta)}{\Gamma(r)}$	
mean	$r\lambda$	
variance	$r\lambda^2$	
median	No simple closed form	
mode	(r-1)λ for r ≥ 1	
mgf	$M_x(t) = \frac{1}{(1-\lambda t)^r}$ for t < $\frac{1}{\lambda}$	
skewness	$^2/_{\sqrt{r}}$	
Kurtosis-excess	$\frac{6}{r}$	

3.8 Logistic Distribution

The **Logistic Distribution** is a continuous probability distribution modeled by a logistic function. It is shaped like the normal distribution with heavier tails (high kurtosis), meaning that it is more prone to producing values that fall far from its mean. It is described by two parameters: the **location parameter, $\boldsymbol{\mu}$** and the **scale parameter, $\boldsymbol{\beta}$.**

Statistic	Formula	Comment
pdf	$P(X = x\|\mu,\beta) = \dfrac{e^{-(x-\mu)/\beta}}{\beta(1+e^{-(x-\mu)/\beta})^2}$; $-\infty < x < \infty;\ -\infty < \mu < \infty$	
cdf	$\dfrac{1}{1+e^{-(x-\mu)/\beta}}$	
mean	μ	
variance	$\dfrac{\pi^2}{3}\beta^2$	
median	μ	
mode	μ	
mgf	$M_x(t) = e^{\mu t}\Gamma(1-\beta t)\Gamma(1+\beta t)$ for t < $\|\frac{1}{\beta}\|$	
skewness	0	
Kurtosis-excess	$\dfrac{6}{5}$	

3.9 Lognormal Distribution

The **Lognormal Distribution** is a continuous probability distribution of a random variable whose logarithm is normally distributed. It is described by two parameters: the **location parameter, μ** and the **shape parameter, σ.**

Statistic	Formula	Comment
pdf	$P(X = x\|\mu, \sigma^2 = \frac{1}{\sqrt{2\pi}\sigma}\frac{e^{-(\log x-\mu)^2/2\sigma^2}}{\text{x}};$ $0 \le x < \infty;\ -\infty < \mu < \infty$	
cdf	$\frac{1}{e^{-(x-\mu)/\beta}}$	
mean	$e^{\mu+\sigma^2/2}$	
variance	$e^{2(\mu+\sigma^2)} - e^{2\mu+\sigma^2}$	
median	e^{μ}	
mode	$e^{\mu-\sigma^2}$	
mgf	$Does\ not\ exist$	
skewness	$(e^{\sigma^2}+2)\sqrt{e^{\sigma^2}-1}$	
Kurtosis-excess	$e^{4\sigma^2} + 2e^{3\sigma^2} + 3e^{2\sigma^2} - 6$	

3.10 Normal Distribution

The **Normal Distribution** or **Gaussian distribution** is a continuous probability distribution often used to describe real-valued random variables that tend to cluster around a single value, the **mean**. The graph, that has a bell shaped form, is also known as the Gaussian function or bell curve. It is described by two parameters: the **location parameter, $\boldsymbol{\mu}$** and the **shape parameter, $\boldsymbol{\sigma}$**. The distribution with $\mu = 0$ and $\sigma = 1$ is called **"standard normal"** distribution.

Statistic	Formula	Comment
pdf	$P(X = x\|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2};$ $-\infty \leq x < \infty;\ -\infty < \mu < \infty$	
cdf	$\frac{1}{2}\left[1 + erf\left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right)\right]$	$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the error function
mean	μ	
variance	σ^2	
median	μ	
mode	μ	
mgf	$exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\}$	
skewness	0	
Kurtosis-excess	0	

3.11 Pareto Distribution

The **Pareto Distribution** is a continuous probability distribution that is often used to describe social, economic, scientific, geophysical, actuarial, and other phenomena. Named after Vilfredo Pareto, it is a power law probability distribution, i.e. in which the frequency of an event varies as a power of some attribute of that event. It is described by two parameters: the scale parameter, α and the shape parameter, β.

Statistic	Formula	Comment
pdf	$P(X = x\|\alpha, \beta) = \dfrac{\beta\alpha^{\beta}}{x^{\beta+1}}$; $\alpha \le x < \infty;\ \alpha, \beta > 0$	
cdf	$1 - \left(\dfrac{\alpha}{x}\right)^{\beta}$	
mean	$\dfrac{\beta\alpha}{\beta - 1}; for\ \beta > 1$	
variance	$\dfrac{\beta\alpha^2}{(\beta - 1)^2(\beta - 2)};\qquad \beta > 2$	
median	$\alpha\sqrt[\beta]{2}$	
mode	α	
mgf	$does\ not\ exist$	
skewness	$\frac{2(1+\beta)}{\beta-3}\sqrt{\frac{\beta-2}{\beta}}$ for $\beta > 3$	
Kurtosis-excess	$\frac{6(\beta^3+\beta^2-6\beta-2)}{\beta(\beta-3)(\beta-4)}$ for $\beta > 4$	

3.12 t Distribution

The **t Distribution** or **Student's t distribution** is a continuous probability distribution that arises when estimating the mean of a normally distributed variable when the sample sizes are too small and the standard deviation is unknown.

The t-distribution is symmetric and bell-shaped, like the normal distribution, with heavier tails (high kurtosis), meaning that it is more prone to producing values that fall far from its mean. It is described by the parameter, ν, the number of degrees of freedom.
The Student's t-distribution is widely used in statistical inference for assessing the statistical significance of the difference between two sample means, the construction of confidence intervals for the difference between two population means, and in linear regression analysis.

Statistic	Formula	Comment
pdf	$P(X = x\|\nu_1, \nu_2) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$; $-\infty < x < \infty;\ \nu = 1,2,\cdots$	
cdf	$= I_x\left(\frac{\nu}{2}, \frac{\nu}{2}\right)$	
mean	$0\ \ for\ \ \nu > 1, otherwise\ undefined$	
variance	$\frac{\nu}{\nu - 2}\ \ for\ \nu > 2$	
median	0	
mode	0	
mgf	Does not exist	
skewness	$0\ for\ \nu > 3$	
Kurtosis-excess	$\frac{6}{\nu - 4}\ \ for\ \nu > 4$	

3.13 Uniform Continuous Distribution

The **Uniform Continuous Distribution** is the continuous probability distribution that models the probability of observing a value on an interval, with all the values on the interval having the same probability of being observed. The distribution is often abbreviated ***U(a,b)***.

The parameters of Uniform Continuous Distribution are ***a*** and ***b***, with ***a*** being the lowest value and ***b*** the largest value of the range.

Statistic	Formula	Comment
pdf	$f(x) = \begin{cases} \frac{1}{b-a}, & for\ a \le x \le b \\ 0, & x \ge 0 \end{cases};$ $b > a;\ x \in [a,b]$	$-\infty < a < b < \infty$
cdf	$\begin{cases} 0 & for\ \ x < a \\ \frac{x-a}{b-a} & for\ a \le x \le b \\ 1 & for\ \ k > b \end{cases}$	
mean	$\frac{a+b}{2}$	
variance	$\frac{(b-a)^2}{12}$	
median	$\frac{a+b}{2}$	
mode	N/A	
mgf	$M_x(t) = {}^{1}/_{N} \sum_{i=1}^{N} e^{it}$	$= \frac{e^{bt} - e^{at}}{t(b-a)}$
skewness	0	
kurtosis	$-\frac{6}{5}$	

3.14 Weibull Distribution

The **Weibull Distribution,** named after the Swedish engineer and mathematician Ernst Hjalmar Waloddi Weibull, is a family of continuous probability distributions that usually models the waiting time until an event, such as failure, happens. It is described by two parameters: **the shape parameter, γ** and the **scale parameter, λ.**

Statistic	Formula	Comment
pdf	$P(X = x\|\gamma, \lambda = \frac{\gamma}{\lambda}\left(\frac{x}{\lambda}\right)^{\gamma-1} e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}$ $0 \le x < \infty; \quad \gamma, \lambda > 0$	
cdf	$1 - e^{-\left(\frac{x}{\lambda}\right)^{\gamma}}$	
mean	$\mu = \lambda\Gamma\left(1 + \frac{1}{\gamma}\right)$	
variance	$\sigma^2 = \lambda^2\left[\Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma^2\left(1 + \frac{1}{\gamma}\right)\right]$	
median	$\lambda(ln(2))^{1/\beta}$ for $\beta \ge 1$	
mode	$\lambda\left(\frac{\gamma-1}{\gamma}\right)^{1/\gamma}$ for $\gamma \ge 1$	
mgf	$\sum_{n=0}^{\infty}\frac{t^n\lambda^n}{n!}\Gamma\left(1 + \frac{n}{\gamma}\right)$	
skewness	$\partial = \frac{\Gamma\left(1+\frac{3}{\gamma}\right)\lambda^3 - 3\mu\sigma^2 - \mu^3}{\sigma^3}$	
Kurtosis-excess	$\frac{\Gamma\left(1 + \frac{4}{k}\right)\lambda^4 - 4\partial\mu\sigma^3 - 4\mu^2\sigma^2 - \mu^4}{\sigma^4} - 3$	